Linear Systems of Equations

CHEN 1703

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Systems of Linear Equations

Any system of linear equations may be written as:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_n \end{aligned}$$

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}$$

Alternatively, $Ax=b$

A Very Simple Example

$$x + y = z \qquad 2x = 5 - 3z \qquad y = 1 + x$$

1. Define the unknown vector, x.
2. Collect unknowns on LHS (ordered the same as the unknown vector x), and collect knowns on RHS.
3. Define the matrix A and RHS vector b.
4. Plug in numbers for A and b, and solve for x. (in Matlab: **`x=A\b`**)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\begin{aligned} x + y - z &= 0 \\ 2x + 3z &= 5 \\ -x + y &= 1 \end{aligned}$$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ -1 & 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$$

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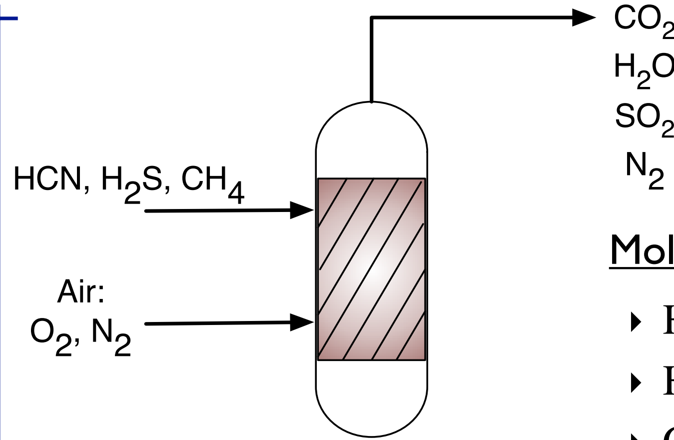
Example: Incineration

Your company needs to eliminate Hydrogen Sulfide and Cyanide toxins that are a byproduct from one of its processes. You are currently using natural gas (primarily CH_4) to incinerate these toxins.

You know the molar flow rates of HCN, H_2S, and CO_2. You are asked to determine how much air is required and how much of each of the products will be produced.



How many unknowns do we have?

Molar Flow rates:

- HCN: 50 kmol/hr
- H_2S: 100 kmol/hr
- CO2: 500 kmol/hr

$$a\,CH_4 + b\,HCN + d\,H_2S + e\,(0.21\,O_2 + 0.79\,N_2) \longrightarrow f\,CO_2 + g\,H_2O + i\,SO_2 + j\,N_2$$

Balance Each Atom:

C balance: $a + b = f$

H balance: $4a + b + 2d = 2g$

O balance: $2 \cdot 0.21e = 2f + g + 2i$

N balance: $b + 2 \cdot 0.79e = 2j$

S balance: $d = i$

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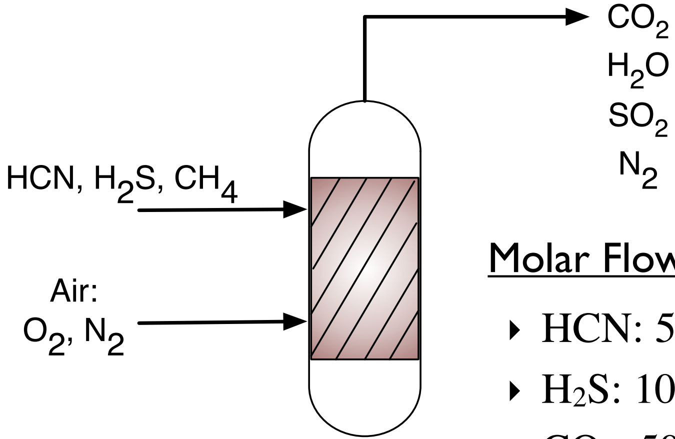

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$$a\,CH_4 + b\,HCN + d\,H_2S + e\,(0.21\,O_2 + 0.79\,N_2) \longrightarrow f\,CO_2 + g\,H_2O + i\,SO_2 + j\,N_2$$

Known: b, d, f
Unknown: a, e, g, i, j

5 unknowns
5 equations!

Goal: write in matrix form, $\mathrm{A}\,x = b$

1. Define the unknown vector, x.
2. Collect unknowns on LHS and knowns on RHS.
3. Define the matrix A and RHS vector b.
4. Plug in numbers for A and b, and solve for x. (in Matlab: **`x=A\b`**)

$$x = \begin{pmatrix} a \\ e \\ g \\ i \\ j \end{pmatrix}$$

Can a System be Solved?

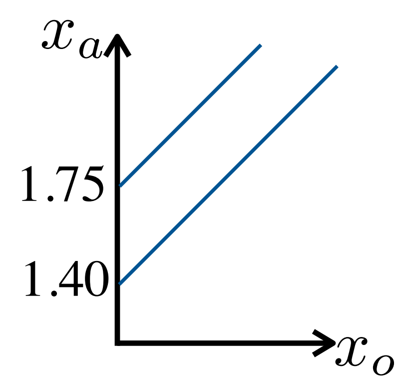
Example:

- Alex buys three apples, two oranges and a pear → \$4.37
- Jenny buys two apples, two oranges → \$2.80
- Rob buys 1 apple, 1 orange → \$1.75

How much does each item cost?

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_a \\ x_o \\ x_p \end{pmatrix} = \begin{pmatrix} 4.37 \\ 2.80 \\ 1.75 \end{pmatrix}$$

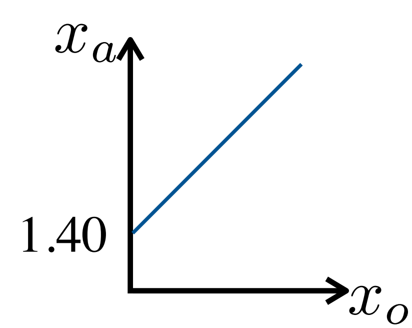
$$\begin{aligned} 2x_a + 2x_o &= 2.80 \\ x_a + x_o &= 1.75 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_a &= -x_o + 1.4 \\ x_a &= -x_o + 1.75 \end{aligned}$$



- Rob buys 1 apple, 1 orange → \$1.40

$$\begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 0 \\ 1 & 1 & 0 \end{bmatrix} \begin{pmatrix} x_a \\ x_o \\ x_p \end{pmatrix} = \begin{pmatrix} 4.37 \\ 2.80 \\ 1.40 \end{pmatrix}$$

$$\begin{aligned} 2x_a + 2x_o &= 2.80 \\ x_a + x_o &= 1.4 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_a &= -x_o + 1.4 \\ x_a &= -x_o + 1.4 \end{aligned}$$



(Potentially) Useful Tools

see “help matfun” for more options.

Matlab Function	Description
det(A)	Calculates the determinant of A. Size of determinant is same as size of A.
cond(A)	Calculates the condition number of A. This gives a measure of the difficulty of solving the system of equations.
eig(A)	Calculate the eigenvalues of A. This can also be used to get the eigenvectors...
rank(A)	Determine the rank of A. If this is less than the size of A, then A cannot be inverted - i.e. it is singular.
lu(A)	Compute the LU factorization of A.

det(A) = 0 if Ax=b cannot be solved.

cond(A) = ∞ if Ax=b cannot be solved.

Number of independent equations in A.