We want to solve
\[ \frac{d^2x}{dt^2} = g - \alpha v, \] (1)
with initial conditions
\[ x(0) = 0, \quad \left. \frac{dx}{dt} \right|_{t=0} = v(0) = 0. \]
Since \( \frac{dx}{dt} = v \), and \( \frac{d^2x}{dt^2} = \frac{dv}{dt} \), (1) can be rewritten as
\[ \frac{dv}{dt} = g - \alpha v, \] (2)
\[ \frac{dx}{dt} = v \] (3)
Since (2) is not a function of \( x \), it can be solved independently from (3) using separation of variables:
\[ \int_{v_0}^{v} \frac{dv}{g - \alpha v} = \int_{t_0}^{t} dt \]
\[ -\frac{1}{\alpha} \left( \ln (av - g) - \ln (\alpha v_0 - g) \right) = t - t_0 \]
\[ \ln \frac{av - g}{\alpha v_0 - g} = \alpha (t_0 - t) \]
\[ v = \frac{g}{\alpha} + \left( v_0 - \frac{g}{\alpha} \right) e^{\alpha(t_0-t)} \] (4)
Now applying the initial conditions \( v_0 = 0 \) and \( t_0 = 0 \), (4) becomes
\[ v = \frac{g}{\alpha} \left( 1 - e^{-\alpha t} \right) \] (5)
We can now substitute (5) into (3) to find
\[ \frac{dx}{dt} = \frac{g}{\alpha} \left( 1 - e^{-\alpha t} \right), \]
which is also separable:
\[ \int_{x_0}^{x} dx = \frac{g}{\alpha} \int_{t_0}^{t} (1 - e^{-\alpha t}) dt, \]
\[ (x - x_0) = \frac{g}{\alpha} \left( t + \frac{1}{\alpha} e^{-\alpha t} - t_0 - \frac{1}{\alpha} e^{-\alpha t_0} \right), \]
\[ x = x_0 + \frac{g}{\alpha^2} \left( \alpha (t - t_0) + e^{-\alpha t} - e^{-\alpha t_0} \right) \] (6)
Now substituting \( t_0 = 0 \) and \( x_0 = 0 \), we find
\[ x = \frac{g}{\alpha^2} \left( \alpha t + e^{-\alpha t} - 1 \right) \] (7)