Problem 1 (14 pts)

Use the following linear system to answer the questions below.

\[
\begin{bmatrix}
3 & 2 & 0 \\
1 & 4 & 2 \\
3 & 5 & 7 \\
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
0 \\
6 \\
9 \\
\end{bmatrix}
\]

The analytic solution to this system is approximately

\[
\begin{bmatrix}
x \\
y \\
z \\
\end{bmatrix}
= 
\begin{bmatrix}
-0.9231 \\
1.385 \\
0.6923 \\
\end{bmatrix}
\]

1. (4 pts) Which of the following are valid forms of the system (circle all that apply)?

- \(3x = -2y, \quad 3z + 2y = 9\)
- \(3xz + 5y - 9 = -7z, \quad z + 4y + 2x = 6\)
- \(x = 6 - 4y - 2z, \quad 3z + 5y + 7z = 0\)

- \(7z + 5y + 3x = 9, \quad 3x/2 + y = 0\)
- \(3x + 2y = 0, \quad z/2 + 2y + z = 3\)
- \(4y + x + 2z = 6, \quad x + 5y/3 + 7z/3 = 3\)

2. (4 pts) Given an iterative solution of \(\begin{bmatrix}
-1 \\
2 \\
0.5 \\
\end{bmatrix}\), what is the \(L_2\) norm of the absolute error in the iterative solution?

3. (6 pts) Show one iteration of the Gauss-Seidel algorithm, using \(\begin{bmatrix}
-1 \\
2 \\
1 \\
\end{bmatrix}\) as an initial guess.
Problem 2 (20 pts)

Use the data in the following table to answer the questions below.

Table 1: Vapor pressure of water as a function of temperature.

<table>
<thead>
<tr>
<th>$T$ (°C)</th>
<th>0</th>
<th>10</th>
<th>20</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p$ (mm Hg)</td>
<td>4.579</td>
<td>9.209</td>
<td>17.535</td>
<td>31.824</td>
<td>55.324</td>
<td>92.51</td>
<td>149.38</td>
</tr>
</tbody>
</table>

1. (4 pts) Estimate $p(33.5)$ using linear interpolation.

2. (4 pts) Show the equations that you need to solve to obtain coefficients for a quadratic interpolant to determine $p(33.5)$.

3. (4 pts) Estimate $\frac{dp}{dT}$ at $T = 25$ °C.
4. (8 pts) One proposed model for $p(T)$ is $\log_{10} p = a + \frac{\beta}{T}$. Show the equations that must be solved to determine $a$ and $\beta$. You can write these using $T_i$, $p_i$, and $N$ (number of points) rather than substituting numbers if you wish. This will save a lot of time.
Problem 3 (12 pts)

Consider the nonlinear equation $y = 9x^2 - 2$, where we want to solve for the value of $x$ that makes $y = 2$ as you answer the questions below.

1. (2 pts) Solve this analytically to determine all of the roots.

2. (4 pts) Which of the following are valid starting guesses for the Regula Falsi algorithm (circle all that apply)?

-1, 1  0, 0.5  -0.5, 0.1  0.6, 0.8

1. (6 pts) Using an initial guess of $x = 2$, perform two iterations of Newton's method and report your estimate of the solution. Then report the residual error given your answer after two iterations.
Problem 4 (14 pts)

Consider the ODE \( \frac{d\phi}{dt} = -\phi^2 \), which is a model for decay of \( \phi \). Assuming that \( \phi(0) = 5 \), determine \( \phi \) at \( t = 0.2 \) using \( \Delta t = 0.1 \) with:

1. (6 pts) Forward Euler

2. (8 pts) Backward Euler
Problem 5 (10 pts)

Consider the boundary value problem

\[ \frac{d^2 T}{d x^2} = S \]

where \( S = e^{-x^2} \) with boundary conditions \( T(x = L) = T_L \) and \( \frac{dT}{dx} \bigg|_{x=0} = \beta_0 \). If we discretize this using finite-difference techniques with \( n = 5 \) points,

1. (8 pts) Show the system of equations that must be solved to determine \( T \). Express your answer in terms of \( \Delta x \), \( T_i \), \( S_i \), \( T_L \) and \( \beta_0 \).

2. (2 pts) What is the value for \( S_3 \) (i.e., \( S(x_3) \)) if \( L = 1? \)