Some Notes on Matrix Operations Relevant to the Momentum and Energy Equations

James C. Sutherland

The "**outer product**" shows up in the momentum equations when written in vector form:

$$\nabla \cdot (\rho \underbrace{\mathbf{v} \otimes \mathbf{v}}_{\text{Outer product}}).$$

In index (Einstein) notation, this is written as:

$$\frac{\partial}{\partial x_i} \left(\rho v_i v_j\right)$$

The outer product takes two vectors and yields a tensor. For example, consider the velocity vector,

$$v_i = \left(\begin{array}{c} v_x \\ v_y \\ v_z \end{array} \right).$$

The outer product of two velocity vectors is written in index notation as

$$v_i v_j = \left[\begin{array}{ccc} v_x v_x & v_x v_y & v_x v_z \\ v_y v_x & v_y v_y & v_y v_z \\ v_z v_x & v_z v_y & v_z v_z \end{array} \right].$$

Contraction of two matrices is written in vector form as $\mathbf{A} : \mathbf{B}$. We see this term in the internal energy and enthalpy equations:

$$\boldsymbol{\tau} : \nabla \mathbf{v},$$

which can be written in index notation as

$$\tau_{ij} \frac{\partial v_i}{\partial x_j},$$

or in summation notation as

$$\sum_{i=1}^{3} \sum_{j=1}^{3} \tau_{ij} \frac{\partial v_i}{\partial x_j}.$$

This is a scalar.