Flux Transformation Matrices

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Consider the $n-1$ dimensional transformations:

$$
\begin{aligned}
(\mathbf{j}) &= [B^{ou}](\mathbf{j}^u), && (1)\\
(\mathbf{j}^u) &= [B^{uo}](\mathbf{j}), && (2)
\end{aligned}
$$

where $[B^{uo}] = [B^{ou}]^{-1}$. In class, we derived an expression for $[B^{ou}]$. Here we will derive the expression for $[B^{uo}]$.

Let's look at the expression for the mass diffusion flux relative to a molar-averaged velocity:

$$
\begin{aligned}
\mathbf{j}_i^u &= \rho_i(\mathbf{u}_i - \mathbf{u}),\\
&= \rho_i(\mathbf{u}_i - \mathbf{v}) + \rho_i(\mathbf{v} - \mathbf{u}),\\
&= \mathbf{j}_i + \rho_i(\mathbf{v} - \mathbf{u}). && (3)
\end{aligned}
$$

Since our goal is to express $\mathbf{j}_i$ in terms of $\mathbf{j}_j^u$, we have rearragned the expression for $\mathbf{j}_i^u$ in terms of $\mathbf{j}_i$ and another term: $\rho_i(\mathbf{v} - \mathbf{u})$. Now let's see if we can use the expression for $\mathbf{j}_i$ to eliminate the $(\mathbf{u} - \mathbf{v})$ term from (3). Recall

$$
\mathbf{j}_i = \rho_i(\mathbf{u}_i - \mathbf{v}). \qquad (4)
$$

This isn't useful to us since it has a $\mathbf{u}_i$ in it. Note, however, that if we sum this, we obtain $\sum_{i=1}^n \mathbf{j}_i = 0$. This is also not useful. Recall we need a term like $\rho_i(\mathbf{u} - \mathbf{v})$. Recall that $\mathbf{u} = \sum_{i=1}^n x_i \mathbf{u}_i$. Look closely at (4). If we multiply by x_i/ω_i then we have

$$
\frac{x_i}{\omega_i}\mathbf{j}_i = \rho x_i(\mathbf{u}_i - \mathbf{v}).
$$

Now if we sum this, we should get a $\mathbf{u}$ from the $x_i\mathbf{u}_i$ term:

$$
\begin{aligned}
\sum_{i=1}^n \frac{x_i}{\omega_i}\mathbf{j}_i &= \rho \sum_{i=1}^n x_i \mathbf{u}_i - \rho\mathbf{v},\\
&= \rho(\mathbf{u} - \mathbf{v}),\\
\frac{1}{\rho}\sum_{i=1}^n \frac{x_i}{\omega_i}\mathbf{j}_i &= \mathbf{u} - \mathbf{v}. && (5)
\end{aligned}
$$

Great! Now we are ready to return to equation (3). Substituting (5) into (3) gives

$$\begin{aligned}
\mathbf{j}_i^u &= \mathbf{j}_i - \frac{\rho_i}{\rho}\sum_{j=1}^{n}\frac{x_j}{\omega_j}\mathbf{j}_j, \\
&= \mathbf{j}_i - \omega_i\left(\sum_{j=1}^{n-1}\frac{x_j}{\omega_j}\mathbf{j}_j + \frac{x_n}{\omega_n}\mathbf{j}_n\right), \\
&= \mathbf{j}_i - \omega_i\left(\sum_{j=1}^{n-1}\frac{x_j}{\omega_j}\mathbf{j}_j - \frac{x_n}{\omega_n}\sum_{j=1}^{n-1}\mathbf{j}_j\right), \\
&= \mathbf{j}_i - \sum_{j=1}^{n-1}\omega_i\left(\frac{x_j}{\omega_j} - \frac{x_n}{\omega_n}\right)\mathbf{j}_j, \\
&= \sum_{j=1}^{n-1}\left(\delta_{ij} - \omega_i\left(\frac{x_j}{\omega_j} - \frac{x_n}{\omega_n}\right)\right)\mathbf{j}_j, \\
(\mathbf{j}^u) &= [B^{uo}](\mathbf{j}),
\end{aligned}$$

with

$$B_{ij}^{uo} = \delta_{ij} - \omega_i\left(\frac{x_j}{\omega_j} - \frac{x_n}{\omega_n}\right).$$

This is equation 1.2.25 in Taylor & Krishna.