Interpolation

ChEn 2450

Given (x_i,y_i), find a function $f(x)$ to interpolate these points.

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Motivation & Concept

Motivation:

- Often we have discrete data (tabulated, from experiments, etc) that we need to interpolate.
- Interpolating functions form the basis for numerical integration and differentiation techniques
  - Used for solving ODEs & PDEs
  - we will cover this later

Concept:

- Choose a polynomial function to fit to the data (connect the dots)
- Solve for the coefficients of the polynomial
- Evaluate the polynomial wherever you want (interpolation)

Properties of air at atmospheric pressure

T K	rho kg/m3	lambda W/(m K)	viscosity N s/m2
100	3.5562	0.0093	7.110e-06
150	2.3364	0.0138	1.034e-05
200	1.7458	0.0181	1.325e-05
250	1.3947	0.0223	1.596e-05
300	1.1614	0.0263	1.846e-05
350	0.9950	0.0300	2.082e-05
400	0.8711	0.0338	2.301e-05
450	0.7750	0.0373	2.507e-05
500	0.6864	0.0407	2.701e-05
550	0.6329	0.0439	2.884e-05
600	0.5804	0.0469	3.058e-05
650	0.5356	0.0497	3.225e-05
700	0.4975	0.0524	3.388e-05
750	0.4643	0.0549	3.546e-05
800	0.4354	0.0573	3.698e-05
850	0.4097	0.0596	3.843e-05
900	0.3868	0.0620	3.981e-05
950	0.3666	0.0643	4.113e-05
1000	0.3482	0.0667	4.244e-05

Incropera & DeWitt, *Fundamentals of Heat and Mass Transfer*, 4th ed.

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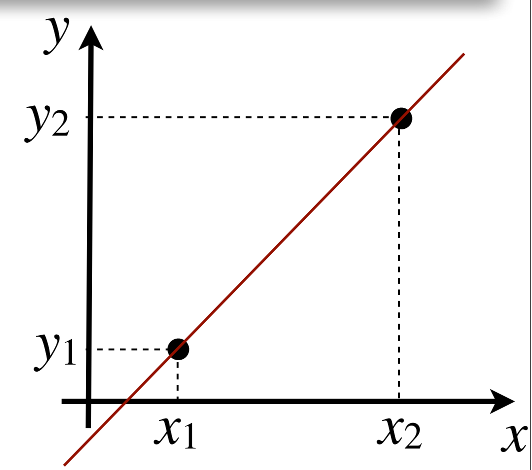
Linear Interpolation

$$y = mx + b \;\rightarrow\; \begin{aligned} y_1 &= mx_1 + b \\ y_2 &= mx_2 + b \end{aligned} \;\rightarrow\; \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \end{bmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

solve for m, b and substitute into the original equation...



Program this into your calculator.

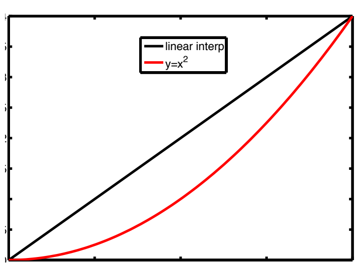
$$y = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1) + y_1$$

Advantages:

- Easy to use (homework, exams)

Disadvantages:

- Not very accurate for nonlinear functions



In MATLAB:

```
yi=interp1(x,y,xi,'linear')
```

- `x` - independent variable entries (vector)
- `y` - dependent variable entries (vector)
- `xi` - value(s) where you want to interpolate
- `yi` - interpolated value(s) at `xi`.

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Polynomial Interpolation

$$p(x) = \sum_{k=0}^{n_p} a_k x^k$$

Given $n+1$ data points, we can fit an n^{th}-degree polynomial.

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_{n+1} & x_{n+1}^2 & \cdots & x_{n+1}^n \end{bmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n+1} \end{pmatrix}$$

Given: (x_i,y_i), solve for a_i

Two steps:

1. Obtain polynomial coefficients by solving the set of linear equations.
2. Evaluate the value of the polynomial at the desired location (x_i)

In MATLAB:

- `p=polyfit(x,y,n)`
  - Forms & solves the above system.
  - requires at least $n+1$ points
  - NOTE: if you supply more than $n+1$ points, then *regression* will be performed (more later).
- `yi=polyval(p,xi)`
  - evaluates polynomial at point(s) given by `xi`.

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Can Apply Polynomial Interpolation "Globally" or "Locally"

Properties of air at atmospheric pressure

T K	rho kg/m3	lambda W/(m K)	viscosity N s/m2
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Incropera & DeWitt, *Fundamentals of Heat and Mass Transfer*, 4th ed.

What is the value of the viscosity at T=412 K?

Polynomial interpolation, n=2

Polynomial interpolation, n=3

Linear interpolation (n=1)

$$p(x) = \sum_{k=0}^{n_p} a_k x^k$$

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Cubic Spline Interpolation

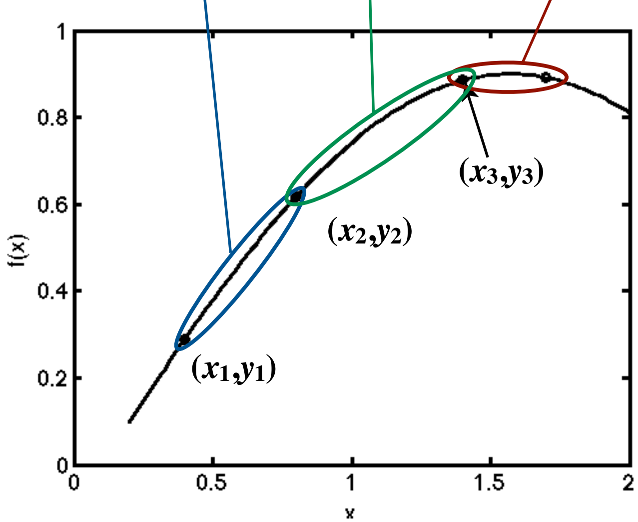
Concept: use cubic polynomial and "hook" them together over a wide range of data...

For n+1 points, we form n splines.
We must specify 4 variables per spline
⇒ **we need 4n equations**.

$$f_3(x) = a_3x^3 + b_3x^2 + c_3x + d_3$$

$$f_2(x) = a_2x^3 + b_2x^2 + c_2x + d_2$$

$$f_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1$$



1. At each data point, the values of adjacent splines must be the same. This applies to all interior points (where two functions meet) ⇒ 2(n-1) constraints.
2. At each point, the *first* derivatives of adjacent splines must be equal (applies to all interior points) ⇒ (n-1) constraints.
3. At each point, the *second* derivative of adjacent splines must be equal (applies to all interior points) ⇒ (n-1) constraints.
4. The first and last splines must pass through the first and last points, respectively ⇒ 2 constraints.
5. The curvature (d^2f / dx^2) must be specified at the end points ⇒ 2 constraints.
   - $d^2f / dx^2 = 0$ ⇒ "natural spline"

4n constraints

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Cubic Spline Interpolation

Advantages:

- Provides a "smooth" interpolant.
- Usually more accurate than linear interpolation.
- Doesn't usually get "wiggly" like higher-order polynomial interpolation can.

Disadvantages:

- Requires a bit more work than linear interpolation to implement.

MATLAB Implementation:

```
yi=interp1(x,y,xi,'spline')
```

- `x` - independent variable entries (vector)
- `y` - dependent variable entries (vector)
- `xi` - value(s) where you want to interpolate
- `yi` - interpolated value(s) at `xi`.

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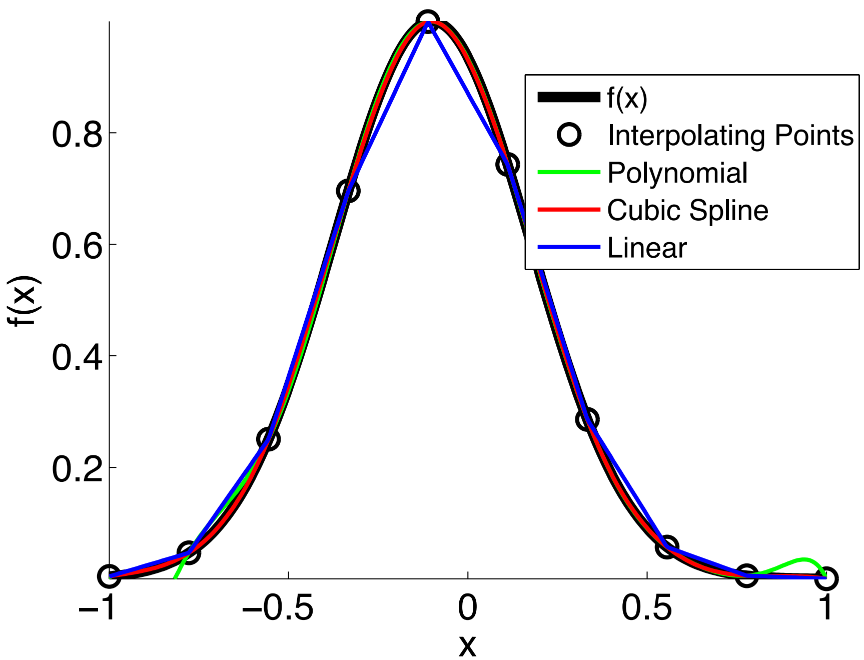
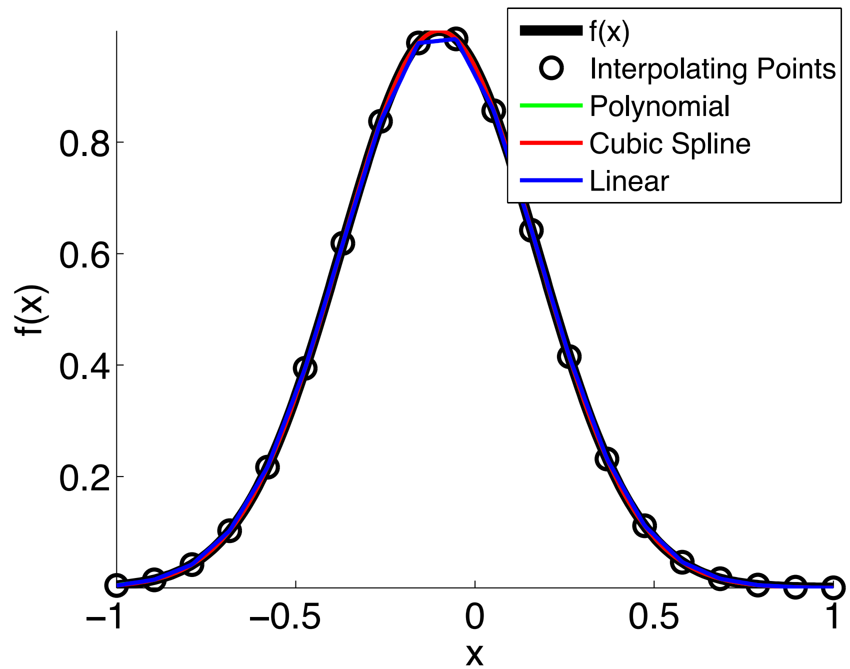
Comparison Between Linear, Spline, & Polynomial Interpolation

$$f(x) = \exp\left(-\frac{(x+b)^2}{c}\right)$$



Increase number of data points.



Spline & polynomial are indistinguishable on this plot.

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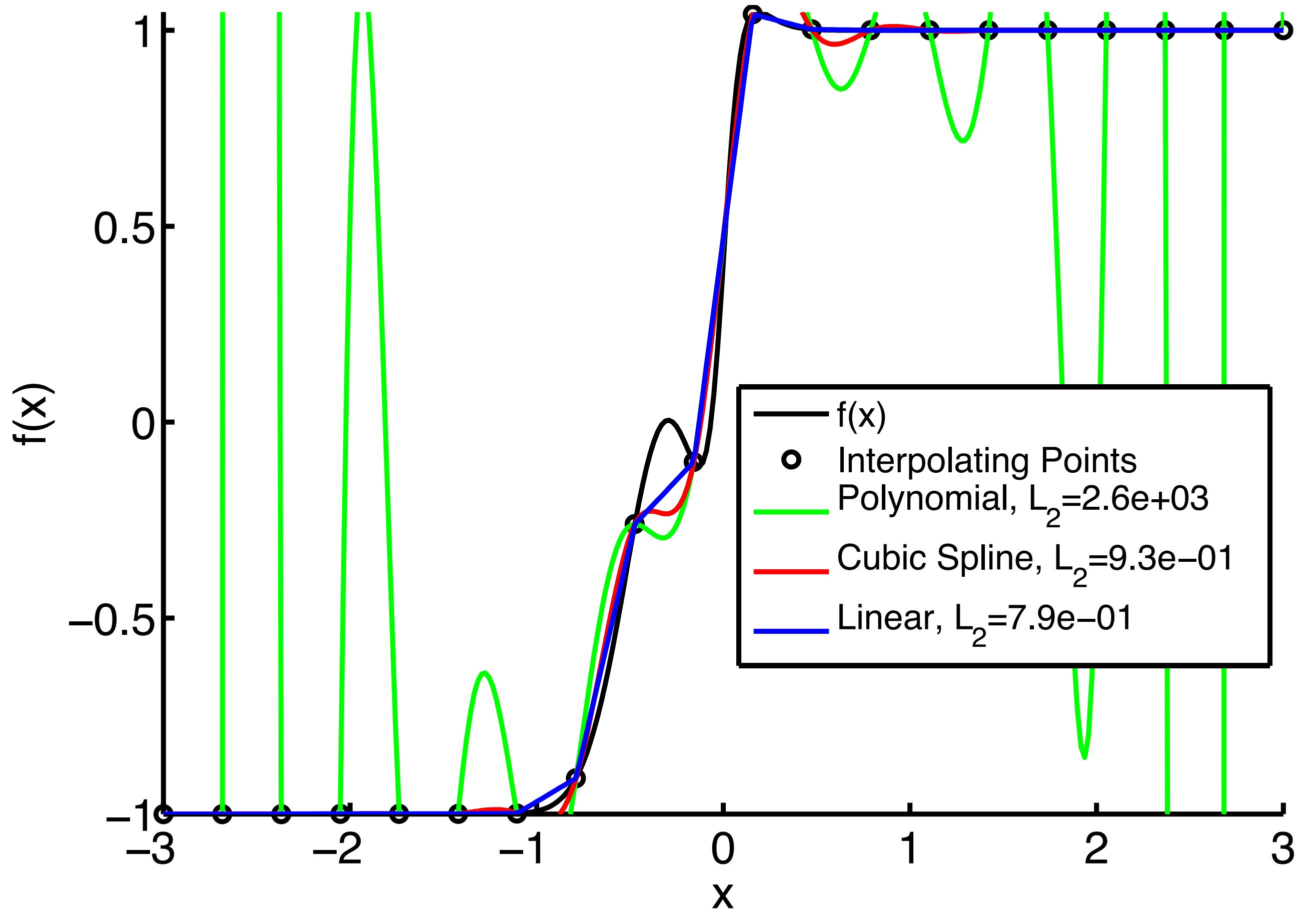
Data points follow $f(x) = \tanh\left(\frac{x}{a}\right) + \exp\left(-\frac{(x+b)^2}{c}\right)$



Polynomial interpolation can be bad if we use high-order polynomials

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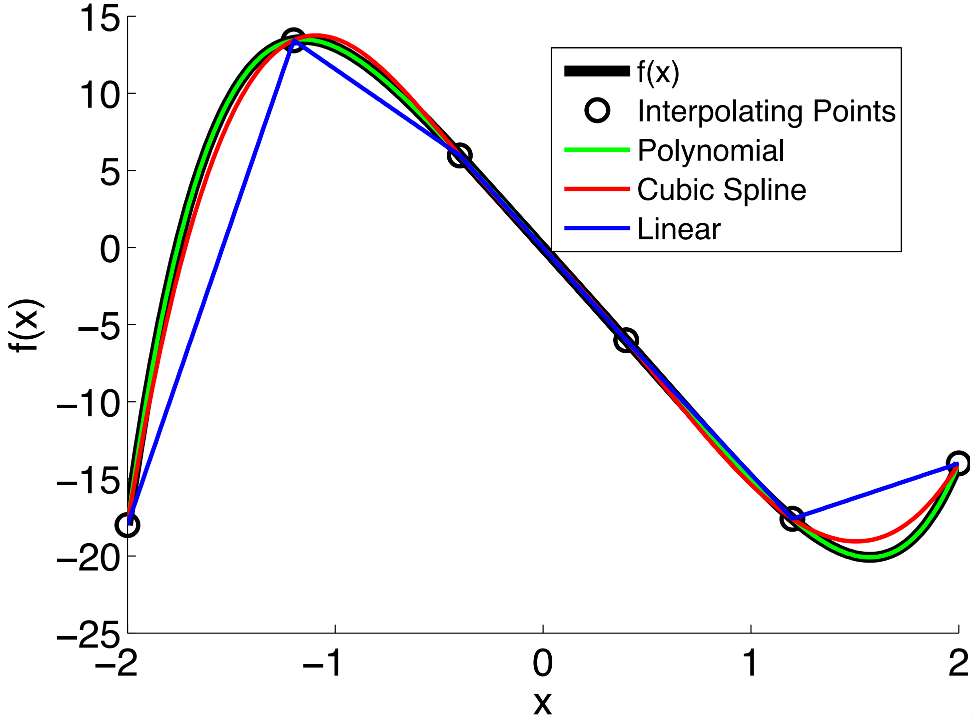
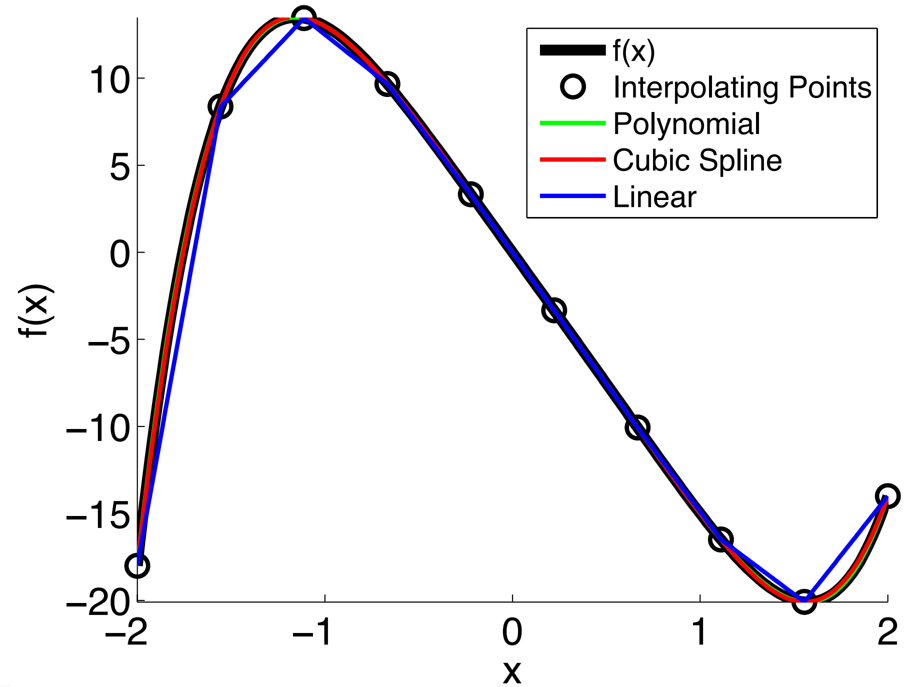
Data points follow $f(x) = x^5 - x^4 - 15x$





- Cubic spline interpolation is usually quite accurate and relatively cost effective.
- Linear interpolation is quick and easy, and may be adequate for well-resolved data.
- Polynomial interpolation can be problematic, unless the underlying data is truly a polynomial!

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Hoffman §4.8.1

2-D Linear Interpolation

Bilinear interpolation

If you have "structured" (tabular) data:

1. Interpolate in one direction (two 1-D interpolations)
2. Interpolate in second direction.

Use this for simple homework assignments, in-class exams, etc.

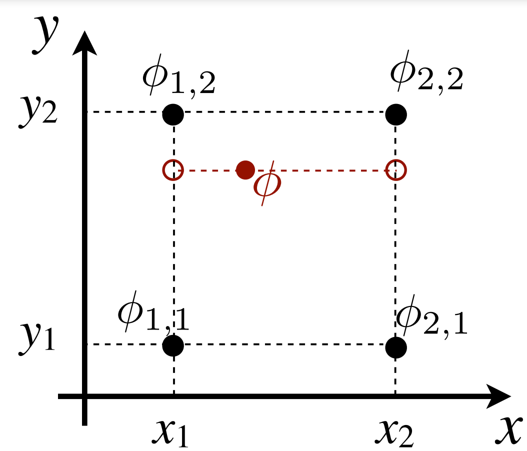
$$\begin{aligned}\phi(x,y) \approx\ & \phi_{1,1}\frac{(x_2-x)(y_2-y)}{(x_2-x_1)(y_2-y_1)} \\ &+ \phi_{2,1}\frac{(x-x_1)(y_2-y)}{(x_2-x_1)(y_2-y_1)} \\ &+ \phi_{1,2}\frac{(x_2-x)(y-y_1)}{(x_2-x_1)(y_2-y_1)} \\ &+ \phi_{2,2}\frac{(x-x_1)(y-y_1)}{(x_2-x_1)(y_2-y_1)}\end{aligned}$$

```
ϕ=interp2(x,y,ϕ,xi,yi,'method')
```

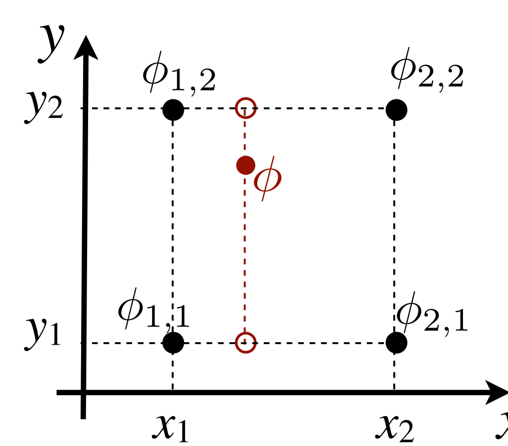
- `'linear'` - 2D linear interpolation (default)
- `'spline'` - 2D spline interpolation

x,y may be **vectors** (matlab assumes tabular form)

ϕ must be a **matrix** (unique ϕ for each x-y pair)



Interpolate y first



Interpolate x first

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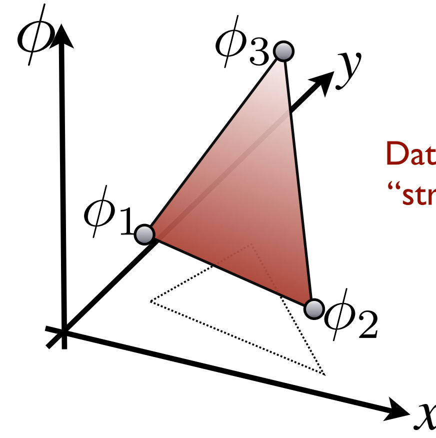
Hoffman §4.8.2

General 2-D Linear Interpolation

Equation of a plane: $\phi = ax + by + c$

$$\begin{aligned}\phi_1 &= ax_1 + by_1 + c\\ \phi_2 &= ax_2 + by_2 + c\\ \phi_3 &= ax_3 + by_3 + c\end{aligned}$$



Data is NOT "structured"

Solve 3 equations for 3 unknowns:

$$a = \frac{\phi_3(y_1-y_2)+\phi_2(y_3-y_1)+\phi_1(y_2-y_3)}{x_3(y_1-y_2)+x_2(y_3-y_1)+x_1(y_2-y_3)},$$

$$b = \frac{\phi_3(x_2-x_1)+\phi_2(x_1-x_3)+\phi_1(x_3-x_2)}{y_3(x_2-x_1)+y_2(x_1-x_3)+y_1(x_3-x_2)},$$

$$c = \frac{\phi_3(x_1y_2-x_2y_1)+\phi_2(x_3y_1-x_1y_3)+\phi_1(x_2y_3-x_3y_2)}{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)}$$

```
ϕ=interp2(x,y,ϕ,xi,yi,'method')
```

- `'linear'` - linear interpolation
- `'spline'` - spline interpolation

x, y, ϕ are **matrices** (unique x,y for each ϕ).

Note: multidimensional higher-order interpolation methods exist (e.g. computer graphics industry)

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