Models for Mass Transfer in “Wall” Bounded Turbulent Flows

References:
• T&K §10.2
Overview

Turbulent boundary layers - why?

- Often in chemical processes, we require mass transfer at interfaces
- Turbulent flow at these interfaces ("walls") increases the rates of mass transfer by increasing gradients.
- Close to walls, molecular viscosity & diffusivity dominate \((L/\eta \to 1)\)
- Away from walls (at large \(Re\)), "turbulent diffusivity" dominates.

"Law" of the wall - describes length & time scales for turbulence near walls.

- Objective: determine the turbulent viscosity (eddy viscosity) from physical arguments.
  Then use \(Sc_{turb}\) to relate \(\nu_{turb}\) and \(k_{turb}\) or \(D_{turb}\).

Model for molecular diffusion

Model for effects of turbulent mixing

\[
\begin{align*}
\dot{j}_{i,\text{net}} &= \dot{j}_i + \dot{j}_{i,\text{turb}} \\
&= -\rho \sum_{j=1}^{n-1} \left( D_{ij} + \delta_{ij} D_{j,\text{turb}} \right) \nabla \bar{\omega}_j \\
&\approx \rho \sum_{j=1}^{n-1} \left( k_{ij} + \delta_{ij} k_{j,\text{turb}} \right) \Delta \bar{\omega}_j
\end{align*}
\]

Mass-transfer coefficient approach
- only consider net effects of mass transfer over some length.
The quest for $\nu_{\text{turb}}$.

**Key parameters** (near the wall):
- $\nu$ (kinematic viscosity) - “momentum diffusivity”
- $\tau_w$ (wall shear stress) - “momentum diffusive flux”

**Wall shear stress:**
$$\tau_w \equiv \rho \nu \left( \frac{\partial \bar{v}_x}{\partial y} \right)_{y=0}$$

**Friction velocity:**
$$\nu_T \equiv \sqrt{\frac{\tau_w}{\rho}}$$

**Viscous length scale:**
$$\delta_\nu \equiv \nu \sqrt{\frac{\rho}{\tau_w}} = \frac{\nu}{\nu_T}$$

Distance from the wall in wall units (“viscous lengths”):
$$y^+ \equiv \frac{y}{\delta_\nu} = \frac{\nu_T y}{\nu}$$

Velocity in wall units:
$$v^+ \equiv \frac{\bar{v}_x}{\nu_T}$$

Friday, April 15, 2011
RANS Momentum equations, constant $\rho, \mu$, no body forces:

$$\frac{\partial}{\partial x_j} (\bar{v}_i \bar{v}_j) + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} - (\nu + \nu_t) \frac{\partial}{\partial x_i} \frac{\partial \bar{v}_j}{\partial x_j} = 0$$

$y, z$ directions have constant mean velocities = 0. $x$-momentum equation is only relevant one.

$$\frac{\partial}{\partial x} (\bar{v}_x \bar{v}_x) + \frac{\partial}{\partial y} (\bar{v}_x \bar{v}_y) + \frac{\partial}{\partial z} (\bar{v}_x \bar{v}_z) + \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x} = (\nu + \nu_t) \left( \frac{\partial^2}{\partial x^2} \bar{v}_x + \frac{\partial^2}{\partial y^2} \bar{v}_x + \frac{\partial^2}{\partial z^2} \bar{v}_x \right)$$

Time-averaged velocity profile.

Near-wall shear stress

$$\frac{1}{\rho} \tau_w = (\nu + \nu_t) \frac{\partial \bar{v}_x}{\partial y}$$

$$= (\nu + \nu_t) \frac{\partial v_x^+}{\partial y^+} \frac{\partial \bar{v}_x}{\partial y^+} \frac{\partial y^+}{\partial \bar{v}_x}$$

$$= (\nu + \nu_t) \frac{\partial v_x^+}{\partial y^+} \frac{\tau_w}{\rho \nu}$$

$$\nu_T = \left( \frac{\partial v_x^+}{\partial y^+} \right)^{-1} - 1$$

Implies this form for $\tau_w$.

$$y^+ \equiv \frac{y}{\delta_v} = \frac{v_T y}{\nu} \Rightarrow \frac{\partial y^+}{\partial y} = \frac{v_T}{\nu}$$

$$v^+ \equiv \frac{\bar{v}_x}{v_T} \Rightarrow \frac{\partial \bar{v}_x}{\partial v_x^+} = v_T$$

$$\frac{\partial y^+}{\partial \bar{v}_x} \frac{\partial \bar{v}_x}{\partial v_x^+} = \frac{v_T^2}{\nu} = \frac{\tau_w}{\rho \nu}$$
The turbulent viscosity

\[ \frac{\nu_T}{\nu} = \left( \frac{\partial v^+}{\partial y^+} \right)^{-1} - 1 \]

Given a wall velocity profile, we may obtain the \( \nu_{turb} \)! Then we can get \( D_{turb} \) from \( Sc_{turb} \).

**von Karman velocity profile (model - one of many)**

Visous sublayer: \( v^+ = y^+ \quad 0 \leq y^+ < 5 \)

Buffer zone: \( v^+ = 5 \ln y^+ - 3.05 \quad 5 \leq y^+ < 30 \)

Turbulent core: \( v^+ = 2.5 \ln y^+ + 5.5 \quad 30 \leq y^+ \)

\( \nu_{turb}/\nu = 0 \quad 0 \leq y^+ < 5 \)

\( \nu_{turb}/\nu = y^+/5 - 1 \quad 5 \leq y^+ < 30 \)

\( \nu_{turb}/\nu = y^+/2.5 - 1 \quad 30 \leq y^+ \)

\( y^+ > 30 \) is typically just called “fully turbulent” where \( \nu \) is ignored.

Note discontinuity in \( \nu_{turb}/\nu \) at \( y^+ = 30 \).
Another way to get $\nu_{\text{turb}}$.

Prandtl’s mixing length hypothesis:

$$\nu_{\text{turb}} = \ell_m \left| \frac{\partial \bar{u}_x}{\partial y} \right|$$

$$\ell^+_m \equiv \ell_m \nu_T = \lambda_p y^+$$

See §10.2.1 in T&K or §7.3.3 in S.B. Pope’s book for derivation...

$$\frac{\nu_T}{\nu} = \left( \frac{\partial v^+}{\partial y^+} \right)^{-1} - 1$$
$$\frac{\partial v^+}{\partial y^+} = \frac{-1 + \sqrt{1 + 4(\ell^+_m)^2}}{2(\ell^+_m)^2}$$
Recap

Model entire turbulence process

- No direct resolution of the flow field (e.g. no DNS, LES, RANS)
- Turbulent mixing $\rightarrow$ increased gradients $\rightarrow$ enhanced diffusion at small scales
- Turbulent mixing affects all species equally
  - Multicomponent effects only present at smallest scales
  - If $L \gg \ell_B$, then molecular mixing is “unimportant” relative to turbulent mixing.
    - Multicomponent effects still exist, but are mixed out rapidly and don’t affect the “large” scales.
  - At walls, multicomponent effects become important since $L/\eta \rightarrow 1$.

Recall:

\[ S_{C_{turb}} = \frac{\nu_{turb}}{D_{turb}} \quad \text{Pr}_{turb} = \frac{c_p \mu_{turb}}{\lambda_{turb}} \]

Therefore, if we prescribe $S_{C_{turb}}, \text{Pr}_{turb}$, then the problem becomes determining $\nu_{turb}$ (or $\mu_{turb}$).

No multicomponent effects for turbulent diffusion terms!